Optimization and Intervals Analysis for Technological Applications

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1. Introduction:

Goal of this research → To provide **methods** (that can be implemented to obtain numerical results) and optimality conditions **for locating set solutions of set-valued optimization problems.**

2. Context

- Applied Mathematics → *Operations Research*: Provides mathematical and quantitative tools aimed to make better decisions (best answer from *feasible set* to a particular problem –*objective* -, under often opposed constraints –e.g.: minimize resources vs maximize benefits in industrial production).
- **General Optimization Problem** → *"Let* $f: X \to \mathbb{R}$. *Given a subset* $M \subset X$, *find the elements* $x \in M$ *in which the largest/smallest value of* f *is attained".*

General Optimization Problem

$$\begin{cases} Min & y = f(x) \\ subject\,to & x \in M \\ f: X \longrightarrow \mathbb{R}, & M \subset X \end{cases}$$

where f is the *objective function* and M the *feasible subset*

Vectorial Optimization Problem

$$(V)\begin{cases} Min & y = f(x) \\ subject\,to & x \in M \subset X \end{cases}$$

where X, Y are topological vector spaces and $f: X \longrightarrow Y$

Multiobjective Optimization Problem

$$(MOP)\begin{cases} Min & y = f(x) \\ subject\,to & x \in M \subset \mathbb{R}^n \\ f: \mathbb{R}^n \longrightarrow \mathbb{R}^p \end{cases}$$

Decision based on a finite number of criteria

Set-Valued Optimization Problem

$$(SVOP)\begin{cases} Min & F(x) \\ subject\,to & x \in M \end{cases}$$

Let Y be a topological linear space ordered by a closed convex cone $K \subset Y$, $M \subset X$ be a nonempty set and $F: M \longrightarrow 2^Y$ be a set-valued map with $F(x) \neq \emptyset$ for each $x \in M$.

3. General Concepts & Background

- Cone K (solid, $K \cap (-K) = \{0\}$) defines partial ordering $\leq_K$
- Y : partially ordered linear space.
- $\wp_0(Y)$: Family of nonempty subsets. If $A \in \wp_0(Y)$:

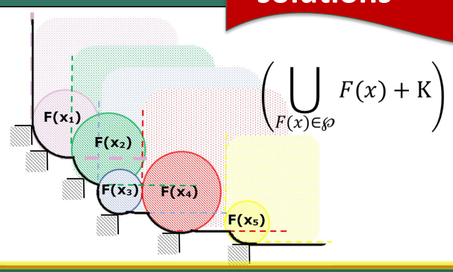
$$\text{Min}\, A = \{y \in A: (y - K) \cap A = \{y\}\}$$

4. Types of criteria of solutions

A. Vector Criterion *(Widely Studied)*

Efficient elements of $F(M) = \bigcup_{x \in M} F(x)$.

$x_0 \in M$ is an **efficient solution** if there exists $y_0 \in F(x_0)$ such that $y_0 \in Min\ F(M)$

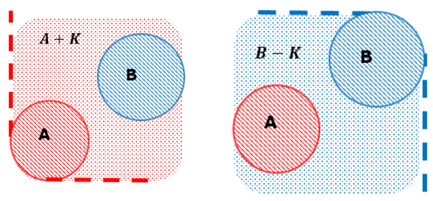


B. Set Criterion *(More recent)*

Efficient elements of $\{F(x): x \in M\}$.

Based on different ***set relations****,* such as:

$$A \leq^{l} B \iff B \subset A + K$$
$$A \leq^{u} B \iff A \subset B - K$$

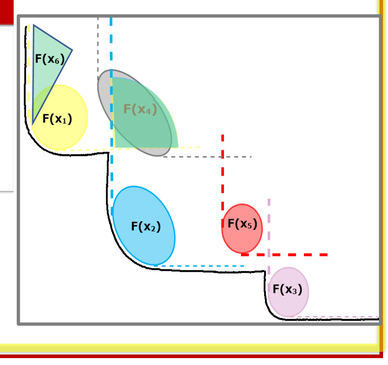


$x_0 \in M$ is an **l(u)-efficient solution** if $x \in M$ and $F(x) \leq^{l(u)} F(x_0)$ imply $F(x_0) \leq^{l(u)} F(x)$

C. Combinations of both, e.g.: Lattice Approach

- ***Case of Study***: *Schrage, C., & Löhne, A. (2013). An algorithm to solve polyhedral convex set optimization problems. Optimization, 62(1), 131-141.*
- 2-steps algorithm:
  - 1. Vectorial Relaxation (Attainment of the **infimum**).
  - 2. **Minimality** with respect to a set relation.

Set Approach	Lattice Approach
$\boldsymbol{F(x_4) \in l - Min(\wp)}$ since $\nexists\, F(x) \in \wp$: $F(x) \leq^{l} F(x_4)$	$\boldsymbol{F(x_4) \notin min \bigcup_{x \in M} F(x)}$ $\boldsymbol{F(x_4) \in l - Min(\wp)}$



- **Working area**: Software implementation for numerical results and real cases analysis.

5. Research and Development opportunities

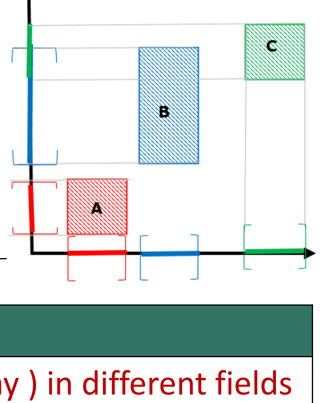
- Set valued optimization problems whose **solutions are defined by set criteria (alone or combined)** is an **expanding field due to its wide applications** in diverse fields: Control theory, Optimization, Economics or Game theory. (Since the set criterion is a natural extension of the minimal solution, vector optimization problems can be also solved by its analysis).

- Different **theoretical works** published during last years **but just a few algorithms proposed without practical implementation** → **1.New implementable algorithms proposals based on set criteria. 2.Numerical results for standard problems in different areas. 3.Algorithms comparison: strengths and weaknesses** → **4.** ***Features to be improved***

How to do it?

- **New pre-order relations definition** (leading to new se relations) would provide different solutions for set-valued optimization problems.

- Linking our results to the **interval analysis** field due to our sets features:

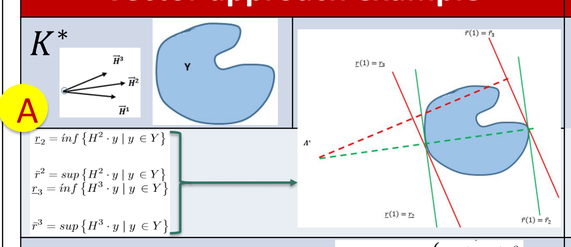
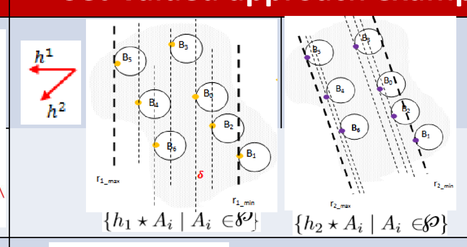
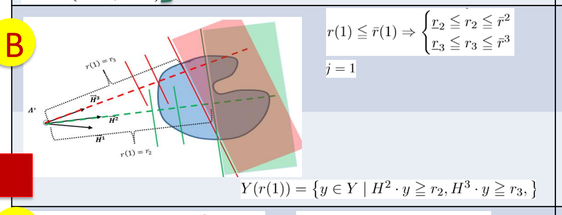
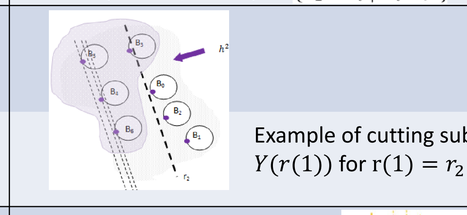
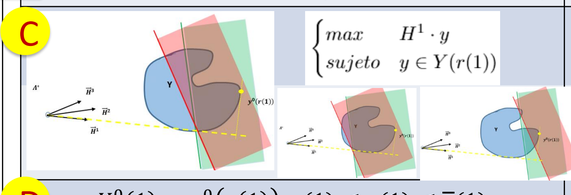
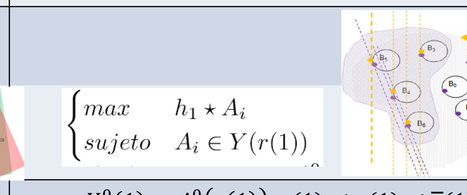


6. Current research fields for this Thesis

- Using non-linear scalarization methods for new algorithms proposals: **Extensions of Gesterwitz's function and support functions.**
- Practical applications of the theory developed *by "Hernández, E., & Rodríguez-Marín, L. (2011). Locating minimal sets using polyhedral cones"* **(**e.g.: **operator ⋆).**

7. Example. Yu's method: Vector vs Set approach

- We extend first method to locate all efficient points of a nonempty set $A \subset \mathbb{R}^n$ introduced by Yu (1974)** in a practical way, by providing the associated algorithms for the vector an set-valued case.

Vector approach example	Set-valued approach example
A 	
B 	 Example of cutting subset $Y(r(1))$ for $r(1) = r_2$
C 	
D $Y^0(1) = y^0(r(1)): \underline{r}(1) \leq r(1) \leq \overline{r}(1)$	$Y^0(1) = A^0(r(1)): \underline{r}(1) \leq r(1) \leq \overline{r}(1)$
E Repeat the steps above for each j value.	Repeat the steps above for each j value.

8. Conclusions

- We work on applications to be used (in a practical way) in different fields such as Industry, Robotics or Games.
- In some cases, set optimization theory is completely applicable to vector optimization theory. Then, finding solutions of set type we can also help to find solutions of vector type.
- New preferences or set-relations might lead to new solution approaches for set-optimization
- Linking set-optimization theory to intervals analysis promise giving applicable results to different areas

** *P.L. Yu, Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives, J. Optim. Theory Appl. 14 (1974) 319–377.*

Unrestricted